The advantage of this method will be still more apparent by integrating* $\cos^3 3\theta\cos\theta d\theta$. Here $\cos^3 3\theta = \frac{1}{8}(x^3+x^{-3})^3 = \frac{1}{8}(x^9+x^{-9})+3(x^3+x^{-3})$.

Multiplying this by $\frac{1}{2}(x+x^{-1})$ we at once have

$$\cos^3 3\theta\cos\theta = \tfrac{1}{8}\cos 10\theta + \tfrac{1}{8}\cos 8\theta + \tfrac{3}{8}\cos 4\theta + \tfrac{3}{8}\cos 2\theta.$$

Whence $\int \cos^3 3\theta\cos\theta d\theta = \frac{1}{80}\sin 10\theta + \frac{1}{64}\sin 8\theta + \frac{3}{32}\sin 4\theta + \frac{3}{16}\sin 2\theta$.

It will be noticed that this form is well adapted for substituting values as limits of integration. For instance if the inferior limit be 0, and the superior limit $\frac{1}{6}\pi$ then $\frac{1}{80}\sin\frac{10}{6}\pi = -\frac{1}{160}\sqrt{3}$; $\frac{1}{64}\sin\frac{8}{6}\pi = -\frac{1}{128}\sqrt{3}$; $\frac{3}{32}\sin\frac{4}{6}\pi = \frac{3}{64}\sqrt{3}$; $\frac{3}{16}\sin 2\theta = \frac{3}{32}\sqrt{3}$.

$$\therefore \int_0^{\frac{1}{6}\pi} \cos^3 3\theta\cos\theta d\theta = \tfrac{81}{640}\sqrt{3}.$$

The reader will have no difficulty in applying the same method to develop $\sin^n\theta$ and then for integrating $\sin^n\theta d\theta$.

It will be observed that when we put $\cos\theta = \frac{1}{2}(x+\frac{1}{x})$ we do not escape the impossible; for this is as much an impossible form as $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ for $x+\frac{1}{x}$ can never be *less* than 2, and $2\cos\theta$ can never be *greater* than 2.

CONCERNING CONICS THROUGH FOUR POINTS.

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The equation of the conic through a_1b_1, a_2b_2, a_3b_3, a_4b_4, and a fifth point x_1y_1 is

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1 & x_1 & y_1 & 1 \\ a_1^2 & a_1b_1 & b_1^2 & a_1 & b_1 & 1 \\ a_2^2 & a_2b_2 & b_2^2 & a_2 & b_2 & 1 \\ a_3^2 & a_3b_3 & b_3^2 & a_3 & b_3 & 1 \\ a_4^2 & a_4b_4 & b_4^2 & a_4 & b_4 & 1 \end{vmatrix} = 0,$$

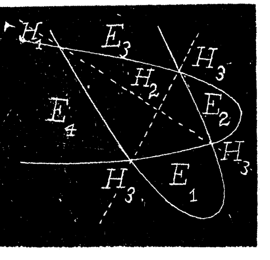
or $Ax^2+2Bxy+Cy^2+2Fx+2Gy+H=0$, where the coefficients A, B, C, ... are of the second degree in x_1 and y_1. The conic is an ellipse, parabola, or hy-

*Professor Waldo first called my attention to this easy method for integrating this particular expression.

perbola according as $AC-B^2$ is greater than, equal to, or less than zero. Through every point of the curve $AC-B^2=0$ (x_1 and y_1 being now general coordinates) may be drawn a parabola also passing through the four given points. Now it is known that through four points two parabolas can be drawn, the parabola being real or imaginary according as one of the four points does not or does lie in the triangle formed by the other three. (See Salmon's Conic Sections, page 153, ex. 1; or C. Smith's Conic Sections, pages 233-4).

Since through every point of each of these two parabolas, a parabola passing through the four given points is possible, the curve $AC-B^2=0$, of the fourth degree, decomposes into these same two parabolas.

Since $AC-B^2$ changes sign when a point crosses the curve, we have determined the locus of those points which with the four given points determine an ellipse (or hyperbola). The curve divides the plane into regions of two kinds, those for which $AC-B^2$ is positive, and those for which $AC-B^2$ is negative. Every point in a region of the first kind determines with the four given points an ellipse; every point of the second kind determines likewise an hyperbola. The points within the region enclosed by the two parabolas determine hyperbolas, since the four points determine a pair of straight lines, passing through this region, and for a pair of straight lines $AC-B^2<O$. Points in the regions marked H (see figure) determine with the four points of intersection of parabolas conics which are hyperbolas; points in the regions marked E determine likewise ellipses.



A particular case of special interest arises when the four points become two pairs of coincident points, and the system becomes that of conics tangent to two given lines at given points. It is easy to show that the two parabolas become coincident. $AC-B^2$ is then a square and cannot change sign. The two tangents constitute one conic of the system and for the present purpose a pair of straight lines is a hyperbola. Hence all conics of the system, with the exceptions of the parabola and the pair of tangent lines, are hyperbolas.

In the above we have supposed points and conics to be real. It is easy to see that the condition for the passing of a real ellipse through four distinct real points is the same as for a real hyperbola. A real parabola can always be drawn through four real points not in the same straight line.